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Algebraization and representation of mereotopological structures

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Abstract. Boolean contact algebras are the abstract counterpart of region–based theories of space, which date back to the early 1920s. In this paper, we survey the development of these algebras and relevant construction and representation theorems.

1 Introduction

The origins of mereotopology go back to the works of [22] on mereology and [21] on the calculus of individuals on the one hand, and, on the other hand, the works of [5], [39], [27], and [45] to use regions instead of points as the basic entity of geometry. In this "pointless geometry", points are now second order definable as sets of regions, similar to the representation of Boolean algebras, where points can be recovered as sets of ultrafilters. An overview of pointless geometry can be found in [18]. Whitehead's addition to the mereological structures of Leśniewski (which were, basically, complete Boolean algebras B without a smallest element) was a "connection" (or "contact") relation C among nonempty regions, which, in its simplest form is a reflexive and symmetric relation satisfying an additional extensionality axiom

$$(1) \qquad (\forall a, b)[((\forall c) aCc \Longrightarrow aCb) \Longrightarrow aPb],$$

where P is Leśniewski's "part-of" relation. Historically, standard (models for) mereotopological structures were collections of regular open sets of topological

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spaces $\langle X, \tau \rangle$ with the *standard (Whiteheadian) contact* among regions defined by

$$uCv \iff \mathrm{cl}(u) \cap \mathrm{cl}(v) \neq \emptyset. \tag{2}$$

The primary example is the collection of all nonempty regular open sets of the Euclidean plane. Other forms of connection have been studied by [4]. In a parallel development, *proximity structures* have been investigated in a topological context since the 1950's. Proximity spaces are relational structures on families of sets that satisfy axioms which to some extent coincide with those for the contact structures mentioned above [16, 26, 37].

Algebraizations of mereotopological structures have been considered for some time and in various ways, see e.g. [2, 3, 19, 21, 31, 36]. Representation results for algebras of regular open sets of locally compact Hausdorff spaces have been given first by [32] and [24][1]. However, their represention does not result in the Whiteheadian contact relation, and we refer the reader to [24, 32] for details.

In the sequel, $\langle B, +, \cdot, {}^*, 0, 1 \rangle$ will denote a Boolean algebra (BA); we will usual identify algebraic structures with their base set. For properties of BAs not explained here we refer the reader to [20]. B^+ is the set of all non-zero elements of B. To avoid trivialities, we assume that all Boolean algebras under discussion have at least four elements. If $M \subseteq B^+$, we call M *dense in* B, if for all $b \in B^+$ there is some $m \in M$ such that $m \leq b$.

2 Binary relations and their algebras

Relations and their algebras have been studied since the latter half of the 19th century by [6], [28] and [33]. [38] gave a first formal introduction to the algebra of relations; his aim was to give an algebraic semantics to first order logic – just as Boolean algebras were an adequate algebraization of classical propositional logic.

Besides the Boolean set–theoretic connectives, natural operations on binary relations on a set U are *composition* and *converse*, defined, respectively, as

$$R \circ S = \{\langle x, y \rangle \in U \times U : (\exists z \in U) xRzSy\}, \tag{3}$$

$$R^{\smile} = \{\langle y, x \rangle : xRy\}. \tag{4}$$

We will usually write xRy for $\langle x, y \rangle \in R$; we also set $R(x) = \{y \in U : xRy\}$.

The *full algebra of relations on* U is the structure $Rel(U) = \langle 2^V, \cup, \cap, -, \emptyset, V, \circ, {}^{\smile}, 1' \rangle$, where $V = U \times U$, and $1'$ is the identity relation. A subset of $Rel(U)$

[1] D. Vakarelov has pointed out in error in [24], and that stronger assumptions are required for the main result to hold.

which is closed under the distinguished operations and contains the constants $\emptyset, V, 1'$ is called an *algebra of binary relations* (BRA). If $\{R_i : i \in I\} \subseteq Rel(U)$, then $\langle R_i \rangle_{i \in I}$ is the subalgebra of $Rel(U)$ generated by $\{R_i : i \in I\}$.

An (abstract) relation algebra (RA) is a structure

$$\langle A, +, \cdot, -, 0, 1, \circ, \breve{}, 1' \rangle$$

of type $\langle 2, 2, 1, 0, 0, 2, 1, 0 \rangle$ which satisfies for all $a, b, c \in A$,

1. $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra (BA).
2. $\langle A, \circ, \breve{}, 1' \rangle$ is an involuted monoid, i.e.
   (a) $\langle A, \circ, 1' \rangle$ is a semigroup with identity $1'$,
   (b) $a\breve{}\,\breve{} = a$, $(a \circ b)\breve{} = b\breve{} \circ a\breve{}$.
3. The following conditions are equivalent:

$$(a \circ b) \cdot c = 0, \ (a\breve{} \circ c) \cdot b = 0, \ (c \circ b\breve{}) \cdot a = 0. \tag{5}$$

The class of all relation algebras will be denoted by **RA**. It can be shown that **RA** is an equational class [38]. Each BRA is an RA, but not vice versa [23]. Many properties of binary relations can be expressed in equational form, for example

$$\begin{aligned} R \text{ is symmetric} &\iff R\breve{} = R, \\ R \text{ is reflexive} &\iff 1' \subseteq R. \\ R \text{ is transitive} &\iff R \circ R \subseteq R, \\ R \text{ is total} &\iff R \circ V = V, \\ R \text{ is one-one} &\iff R \circ R\breve{} \subseteq 1'. \end{aligned}$$

The following construction will be useful in later Sections: Let A be an RA, and suppose that $a, b \in A$. Even though the equation $a \circ x = b$ does not always have a solution, there is an element $a \setminus_{\text{res}} b$, called the *residual* of b by a, such that

$$a \circ x \leq b \iff x \leq a \setminus_{\text{res}} b.$$

The residual can be expressed as an RA term in a and b by

$$a \setminus_{\text{res}} b = -(a\breve{} \circ -b). \tag{6}$$

If a is symmetric, then $a \setminus_{\text{res}} a$ reduces to $-(a \circ -a)$; we will denote this expression by $\text{res}(a)$. It has been shown by [30] that $a \setminus_{\text{res}} a$ is reflexive and transitive. For $R, S \in Rel(U)$, the residual is given by the condition

$$x(R \setminus_{\text{res}} S)y \iff R\breve{}(x) \subseteq S\breve{}(y). \tag{7}$$

The expressiveness of RA logic turns out to be equipollent to a fragment of first order logic [40]:

Proposition 1. *If $R_0, \dots, R_k \in Rel(U)$, then $\langle R_0, \dots, R_k \rangle$ is the set of all binary relations on U which are definable in the (language of the) relational structure $\langle U, R_0, \dots, R_k \rangle$ by first order formulas using at most three variables, two of which are free.*

3 Topological spaces

For any notion not explained here, we invite the reader to consult [17]. We will denote topological spaces by $\langle X, \tau \rangle$, where τ is the topology on X; for $u \subseteq X$, we let $\mathrm{cl}_\tau(u)$ be the τ-closure of u, and $\mathrm{int}_\tau(u)$ its τ-interior. If τ is understood, we will just speak of X as a topological space, and drop the subscripts from the operators. Unless otherwise indicated, we assume for the rest of the paper that

$$\text{All topological spaces considered are } T_1,$$

i.e. that singletons are closed. For $u \in \tau$, we let $\partial(u) = \mathrm{cl}(u) \setminus u$ be the *boundary* of u. If $u, v \in \tau$, then u and v are called *separated*, if $\mathrm{cl}(u) \cap v = u \cap \mathrm{cl}(v) = \emptyset$. A non–empty open set u is called *connected* if it is not the union of two separated nonempty open sets.

$u \subseteq X$ is called *regular open* if $u = \mathrm{int}(\mathrm{cl}(u))$, and *regular closed*, if $u = \mathrm{cl}(\mathrm{int}(u))$. The set complement of a regular open set is regular closed and vice versa. $\mathrm{RegCl}(\langle X, \tau \rangle)$ is the collection of regular closed sets, and $\mathrm{RegOp}\langle X, \tau \rangle)$ the collection of regular open sets; we will sometimes just write $\mathrm{RegCl}(X)$ or $\mathrm{RegCl}(\tau)$ (respectively, $\mathrm{RegOp}(X)$ or $\mathrm{RegOp}(\tau)$) if no confusion can arise. It is well known that $\mathrm{RegCl}(X)$ is a complete Boolean algebra with the operations $a + b = a \cup b$, $a \cdot b = \mathrm{cl}(\mathrm{int}(a \cap b)))$, and $a^* = \mathrm{cl}(X \setminus a)$. Note that we can have $a \cdot b = 0$, while $a \cap b \neq \emptyset$. Similarly, $\mathrm{RegOp}(X)$ is a Boolean algebra with the operations $a + b = \mathrm{int}(\mathrm{cl}(a \cup b)),\ a \cdot b = a \cap b,\ a^* = \mathrm{int}(X \setminus a)$.

A topological space is called *regular* if for each $x \in X$ and each nonempty closed set A not containing x, there are disjoint open sets u, v such that $x \in u$ and $a \subseteq v$. It is well known that X is regular, if and only if for each non–empty $u \in \tau$ and each $x \in u$ there is some $v \in \tau$ such that $x \in v \subseteq \mathrm{cl}(v) \subseteq u$. We call X *weakly regular* if it is semiregular (i.e. it has a basis of regular opens sets) and for each non–empty $u \in \tau$ there is some non–empty $v \in \tau$ such that $\mathrm{cl}(v) \subseteq u$. Weak regularity may be called a "pointless version" of regularity, and each regular space is weakly regular.

A topological space X is called *normal*, if any two disjoint closed sets can be separated by disjoint open sets. X is called *κ–normal*, if any two disjoint regular closed sets can be separated by disjoint open sets. Then,

$$X \text{ is normal} \Longrightarrow X \text{ is } \kappa\text{–normal} \Longrightarrow X \text{ is regular} \Longrightarrow X \text{ is weakly regular} \Longrightarrow X \text{ is semiregular},$$

and none of the implications can be reversed: [34] gives an example of a κ-normal space which is not normal, and of a regular space which is not κ-normal. [13] exhibit a weakly regular T_1 space which is not T_2, and thus, it is not regular. Finally, [35, Example 60] provide a connected, semiregular space which is not weakly regular.

4 Proximities

Suppose that $a, b \subseteq X$. The intuitive meaning of a proximity δ is that $a \delta b$ holds, when a is close to b in some sense. Proximities were introduced by [16] in the early 1950s, and they show a remarkable likeness to the Boolean connection algebras to be discussed below. It is therefore surprising that in mereotopology little attention has been paid to these structures. The main source on proximity spaces is the monograph by [26].

Formally, a binary relation δ on the powerset of a set X is called a *proximity*, if it satisfies the following axioms for $a, b, c \subseteq X$:

P1. If $a \cap b \neq \emptyset$ then $a \delta b$.
P2. If $a \delta b$ then $a, b \neq \emptyset$.
P3. δ is symmetric.
P4. $a \delta (b \cup c)$ iff $a \delta b$ or $a \delta c$.
P5. If $a(-\delta)b$ then $a(-\delta)c$ and $b(-\delta)c^*$ for some $c \subseteq a$.

A proximity is called *separated* if it satisfies

P_{sep} $\{x\}\delta\{y\}$ implies $x = y$.

The pair $\langle 2^X, \delta \rangle$ is called a *proximity space*. Each proximity space determines a topology on X in the following way: Define an operator cl on 2^X by

(8) $$\mathrm{cl}(a) = \{x \in X : \{x\}\delta a\}.$$

Now,

Proposition 2. *[26]*

1. *The operation of* (8) *defines the closure operator of a topology $\tau(\delta)$ on X which is not necessarily T_1.*
2. *$\langle 2^X, \tau(\delta) \rangle$ is a completely regular space. If δ is separated, then $\langle 2^X, \tau(\delta) \rangle$ is also a T_1 space.*
3. *$a \delta b$ iff $\mathrm{cl}(a) \delta \, \mathrm{cl}(b)$.*

A proximity which is relevant to our investigation is the *standard proximity* on X [26]: Let $\langle X, \tau \rangle$ be a normal (T_1) space and define

(9) $$a \delta b \iff \mathrm{cl}(a) \cap \mathrm{cl}(b) \neq \emptyset.$$

Observe that δ is separated, since X is T_1 and thus, singletons are closed.

5 Boolean contact algebras

5.1 Definition and first properties

A standard model of regions are the regular closed sets of a regular T_1 topological space $\langle X, \tau \rangle$ with a relation C defined on $\text{RegCl}(X)$ by

$$uCv \iff u \cap v \neq \emptyset. \tag{10}$$

C will be called the *standard (topological) contact* on X. Even though we will be concerned mostly with $\text{RegCl}(X)$, it is worthy to mention that the structure $\langle \text{RegOp}(X), C_{\text{RegOp}} \rangle$ with C_{RegOp} defined by

$$uC_{\text{RegOp}}v \iff \text{cl}(u) \cap \text{cl}(v) \neq \emptyset \tag{11}$$

is isomorphic to $\langle \text{RegCl}(X), C \rangle$.

The concept of a *Boolean contact algebra* (BCA), as defined below, is an attempt to model standard topological contact; a variant, *Boolean connection algebras*, were introduced by [36] in order to capture the algebraic properties of the RCC system, which we will discuss below. A slightly different system was given by [44] to capture proximity spaces. The notion of BCA is, in a way, the common part of these systems. More formally, a binary relation C on a Boolean algebra B is called a *contact relation* if it satisfies

C0. $aCb \Longrightarrow a, b \neq 0$.
C1. $a \neq 0 \Longrightarrow aCa$.
C2. C is symmetric.
C3. aCb and $b \leq c \Longrightarrow aCc$ (The compatibility axiom).
C4. $aC(b + c) \Longrightarrow aCb$ or aCc (The sum axiom).
C5. $C(a) = C(b) \Longrightarrow a = b$ (The extensionality axiom).

As shown in [43], in the presence of the other axioms we can replace C5 by

C5'. If $a \not\leq b$, there is some $c \in B$ such that aCc and $c(-C)b$.

Furthermore, it is known from [11] that we may also replace C5 by

C5". $\text{res}(C)$ is antisymmetric.

Since $\text{res}(C)$ is always reflexive and transitive, C5" says that $\text{res}(C)$ is a partial order which we denote by P (for "part of").

If C is a contact relation on B, the pair $\langle B, C \rangle$ will be called a *Boolean contact algebra* (BCA). We will consider the following additional properties of C:

C6. $a(-C)b \Longrightarrow (\exists c)[a(-C)c$ and $-c(-C)b]$ (The interpolation axiom).

C7. $a \neq 0, 1 \Longrightarrow aCa^*$ (The connection axiom).

In these axioms (and henceforth) we denote the complement in the underlying Boolean algebra by *, and the set complement in $2^{B \times B}$ by $-$.

Some relations which are relationally definable from C, and which will be used in the sequel, are given in Table 1. In a way, these are the two–dimensional

Table 1. Some C–definable relations

(12)	$P = -(C \circ -C),$	part of
(13)	$PP = P \cap -1'.$	proper part of
(14)	$O = P^{\smile} \circ P$	overlap
(15)	$PO = O \cap -(P \cup P^{\smile})$	partial overlap
(16)	$EC = C \cap -O$	external contact
(17)	$TPP = PP \cap (EC \circ EC)$	tangential proper part
(18)	$NTPP = PP \cap -TPP$	non–tangential proper part
(19)	$DC = -C$	disconnected

version of Allen's interval relations [1], see [9] for details. Additionally, we define

(20) $$ECD = -[(EC \circ EC^{\smile}) \cup (EC^{\smile} \circ EC)],$$

(21) $$ECN = EC \cap -ECD.$$

Observe that $aECDb \Longleftrightarrow b = a^*$ [10]. Furthermore, C3 and C5 imply that P is the Boolean partial order $\leq$ on B^+, and that therefore

(22) $$aOb \Longleftrightarrow a \cdot b \neq 0.$$

For later use we mention some properties of these relations:

Lemma 1. *[10]*

1. *$ECN = TPP \circ ECD$, i.e. $xECNz \iff xTPPz^*$.*
2. *If $xDCz$, then $xTPP(x + z)$.*
3. *$xNTPPz$ and $yNTPPz \iff (x + y)NTPPz$.*
4. *If $xNTPPz$, then $(x^* \cdot z)TPPz$.*
5. *$TPP \subseteq ECN \circ ECN$.*
6. *$xTPPy \Longleftrightarrow xECNy^*$. .*

The following result shows that contact relations on finite BAs, or, more generally, on finite–cofinite algebras, are not very illuminating. The second part demonstrates an important relationship between relational properties of C and a property of B:

Proposition 3. *1. [15] O is the smallest contact relation on B. If B is a finite–cofinite algebra, then O is the only contact relation on B.*
2. [12] If C satisfies C7, then B is atomless.

Let us consider the expressiveness of BCAs of topological properties; as an example we will show that in regular T_1 spaces, the standard topological contact (11) on $\mathrm{RegOp}(X)$ is strong enough to express topological connectivity of regular open sets.

Proposition 4. *[15] Let $B = \mathrm{RegOp}(X)$ for some regular T_1 space X. Let $u \in B \setminus \{\emptyset, X\}$. Then,*

$$u \text{ is not connected } \iff (\exists s \in B)[\emptyset \neq s \subsetneq u \text{ and } TPP^{\smile}(s) \subseteq TPP^{\smile}(u).$$

Corollary 5 *Let $\langle B, C\rangle$ be as above. Then, there is a relation $T \subseteq 1'$ in the RA generated by C such that for all $u \in B^+$, u is not connected iff uTu.*

Proof. Applying (7) and Proposition 4 we obtain

$$u \text{ is disconnected } \iff u(PP^{\smile} \circ (TPP \diagdown_{\text{res}} TPP))u. \tag{23}$$

We write $\mathtt{disc}(u)$ if u fulfills condition (23).

An *RCC algebra* is a BCA which satisfies C7. These correspond to the Region Connection Calculus of [31] which has received some prominence in qualitative spatial reasoning. If B satisfies C0 – C6 we call it a *proximity BCA* (PBCA). These algebras were introduced by [44] in relation with proximity spaces.

An alternative axiomatization of BCAs can be given via the $NTTP$ relation. This relation is known under various names, such as "well inside", "well below", "interior parthood", or "deep inclusion". For better readability, and in keeping with the tradition, will write $\ll$ instead of $NTPP$.

First, observe that

$$a \ll b \iff a(-C)b^*. \tag{24}$$

Now, consider the following statements:

($\ll$0). $0 \ll a$.
($\ll$1). $a \ll b \Longrightarrow a \leq b$.

($\ll$2). $a \ll b \Longrightarrow b^* \ll a^*$.
($\ll$3). $a \ll b$ and $b \leq c \Longrightarrow a \ll c$.
($\ll$4). $a \ll b$ and $a \ll c \Longrightarrow a \ll b \cdot c$.
($\ll$5). $a \not\leq b \Longrightarrow (\exists z)[z \ll a$ and $z \not\ll b]$.
($\ll$6). $a \ll c \Longrightarrow (\exists b \in B)[a \ll b \ll c]$.
($\ll$7). $a \ll a \Longrightarrow a \in \{0, 1\}$.

Given a contact relation C, we let $x \ll_C y \iff x(-C)y^*$, and, given $\ll$, we let $xC_{\ll}y \iff x(- \ll)y^*$. Then, $C_{\ll_C} = C$, and $\ll_{C_{\ll}} = \ll$, so we just write C and $\ll$ in the sequel, assuming these definitions. The detailed relationship among the axioms is as follows:

$$(\ll 0) \iff \text{C0}. \tag{25}$$
$$(\ll 1) \Longrightarrow \text{C1}. \tag{26}$$
$$(\ll 1) \Longleftarrow \text{C1,C3}. \tag{27}$$
$$(\ll 2) \iff \text{C2}. \tag{28}$$
$$(\ll 3) \iff \text{C3}. \tag{29}$$
$$(\ll 4) \iff \text{C4}. \tag{30}$$
$$(\ll 5) \iff \text{C5}. \tag{31}$$
$$(\ll 6) \iff \text{C6}. \tag{32}$$
$$(\ll 7) \iff \text{C7}. \tag{33}$$

The $\ll$ relation was used by [24] in connection with continuous lattice, and by [43] in connection with proximity spaces.

5.2 Constructions of BCAs

We first make sure that full regular open algebras are models for the various classes of BCAs; the following results show that the various axioms of BCAs are intimately related to topological properties.

Proposition 6. *[13] Suppose that $\langle X, \tau \rangle$ is a topological space and C the standard topological contact on $B = \text{RegCl}(X)$. Then,*

1. *C satisfies C0 – C4.*
2. *C satisfies C5 if and only if X is weakly regular.*
3. *C satisfies C6 if and only if X is κ–normal.*
4. *C satisfies C7 if and only if X is connected.*

Corollary 7 *Let $B = \text{RegCl}(X)$ for some topological space X. Then,*

1. B is a BCA if and only if X is weakly regular.
2. B is an RBCA if and only if X is weakly regular and connected.
3. B is PBCA if and only if X is weakly regular and κ–normal.

As the axioms for BCAs are first order, we may expect them to have non-standard models in the sense of Section 5.1. In particular, they must have countable models which, clearly, are not complete Boolean algebras. We will give two examples of countable BCAs: The first one, the interval algebra of a dense linear order could be called a one–dimensional BCA. We then use this structure to build a rather unusual BCA, where every region is full of holes. This part is taken from [14], where the proofs can be found.

First, we recall the definition of an interval algebra [20]. Let L be a dense linear order with smallest element m. Suppose that ∞ is a symbol not in L, and set $L^+ = L \cup \{\infty\}$ with $x \lneq \infty$ for all $x \in L$. An *interval of* L is a set of the form $[s, t) = \{u \in L : s \leq u \lneq t\}$. $IntAlg(L)$ is the collection of all finite unions of intervals

$$[x_0^0, x_0^1) \cup [x_1^0, x_1^1) \cup \ldots \cup [x_{t(x)}^0, x_{t(x)}^1), \tag{34}$$

together with the empty set. It is well known that $IntAlg(L)$ is a Boolean algebra [20, see p.10], called *the interval algebra of L.*

Each nonzero $x \in IntAlg(L)$ can be written in the form (34) in such a way that $x_j^i \in L^+$, $x_j^0 < x_j^1 < x_{j+1}^0$, and the intervals $[x_j^0, x_j^1)$ are pairwise disjoint. The representation of x in this form is unique, and we call it the *standard representation.* In the sequel, we shall assume that all elements of $IntAlg(L)$ are in standard representation.

For each $x \in IntAlg(L) \setminus \{\emptyset\}$, we let $\mathrm{rel}(x) = \{x_j^0 : j \leq t(x)\} \cup \{x_j^1 : j \leq t(x)\}$ be the set of *relevant points* of x. Next, we define a binary relation C on $IntAlg(L)$ by

$$xCy \iff x \cdot y > 0 \text{ or } \mathrm{rel}(x) \cap \mathrm{rel}(y) \neq \emptyset. \tag{35}$$

The following now is not hard to show:

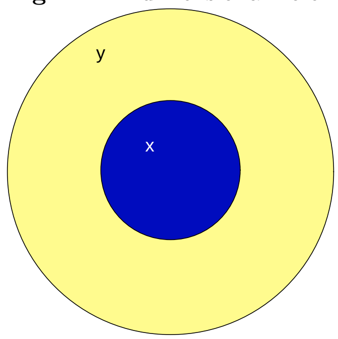
Proposition 8. *[14] $\langle B, C \rangle$ satisfies C0 – C7.*

Our next example is a more exotic one, which, together with a Representation Theorem given below, has interesting topological consequences. [25] defines a "hole relation" H by

$$xHy \iff xECy,\ x + y \neq 1,\ (\forall z)[xECz \Longrightarrow yOz]. \tag{36}$$

xHy is read as "x is a hole of y". In Figure 1, x is a hole of y; Figure 2 shows a hole x of a sphere y in 3–space.

Fig. 1. A 2–dimensional hole



Lemma 2. *If* $x \neq 0,\ x+y \neq 1,\ x \cdot y = 0$*, then*

$$xHy \Longleftrightarrow xNTPP(x+y). \tag{37}$$

Proof. "$\Longrightarrow$": Assume $xTPP(x+y)$, i.e. $xECN(x^* \cdot y^*)$. Since xHy, we have $(x^* \cdot y^*)Oy$, a contradiction.

"$\Longleftarrow$": Since $xNTPP(x+y)$, we have $xDC(x^* \cdot y^*)$. Let $sECNx$, and assume $s \cdot y = 0$, i.e. $s \leq y^*$. Then, $s \leq x^*$ implies that $s \leq x^* \cdot y^*$, and hence, $xDCs$, contradicting $xECNs$.

A region y is called *solid*, if it does not have any holes, i.e. if $H^{\smile}(y) = \emptyset$. In other words,

(38)
$$y \text{ is solid } \Longleftrightarrow (\forall x \in B^+)[xECNy \Longrightarrow (\exists z \in B^+)(xECNz \text{ and } z \cdot y = 0)].$$

Lemma 3. *Suppose that x is a hole of y. Then,*

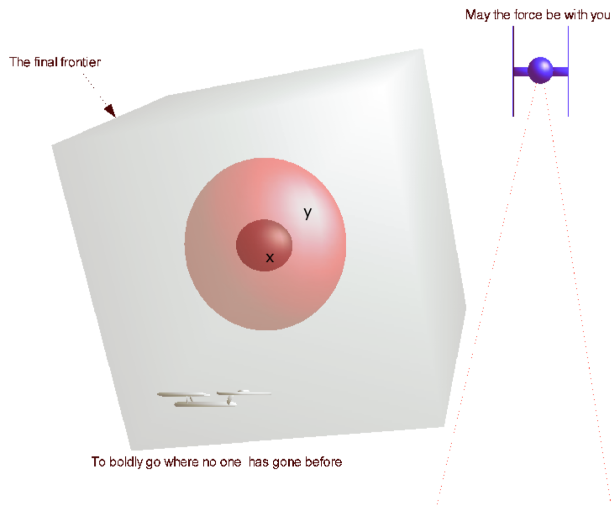
1. *$(x+y)^*$ is also a hole of y.*
2. *y^* is disconnected in the sense of* (23)*.*

Proof. 1. was shown in [25]. For 2. we first show

$$\neg\, \mathtt{disc}(u) \Longrightarrow (\forall s,t \in B^+)[s+t = u \Longrightarrow sCt]. \tag{39}$$

Proof. Suppose that $\neg\, \mathtt{disc}(u)$ and assume that there are $s,t \in B^+$ such that $s+t = u$ and $sDCt$. Then, there is some $z \in B^+$ such that $zTPPs$ and $zNTPPu$. Now, $zTPPs$ implies that $zECNs^*$ by Lemma 1(1), and since $s^* = t + u^*$ we have zCt or zCu^* by C4. The first case contradicts $sDCt$, and the second case contradicts $zNTPPu$.

Fig. 2. A 3–dimensional hole



Since $y^* = x + x^* \cdot y^*$, all that is left to show is that $x(-C)(x^* \cdot y^*)$, i.e. $xNTTP(x + y)$. But this follows immediately from (37).

Mormann conjectured that every RCC algebra contained solid proper regions. Below, we will exhibit an RCC algebra in which every proper region has infinitely many holes. Our example will involve the construction of two Boolean algebras, of which the first algebra serves as an index for a family of interval algebras, from which we will construct our desired model. As a preparation, we characterize the solid regions in an interval algebra. The result shows that not all intervals are solid, which is somewhat surprising, and one wonders whether the terminology of "hole" and "solid" are adequate for BCAs in which some regions have "absolute boundaries" (see Figure 3):

Lemma 4. *Suppose that $B = IntAlg(L)$, and $m = \min L$.*

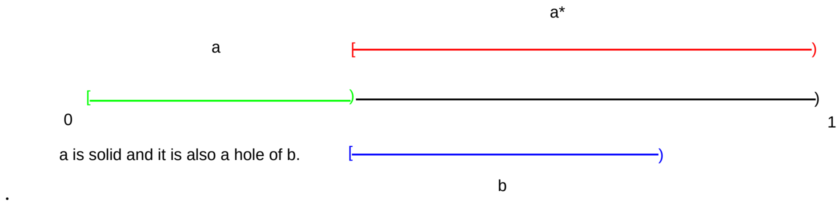
1. *$x \in B^+$ is solid if and only if x has the form $[m, s)$ or $[t, \infty)$ or $[m, s) \cup [t, \infty)$ with $s \lneq t$.*
2. *If xHy, then there is some $z \in B^+$ such that zHy and z is an interval.*

The strategy is to let $\mathbb{Q}_n$ be the rational interval $[n, n+1)$ and $L_n = IntAlg(\mathbb{Q}_n))$. If x is a solid of L_0, then

1. Replicate x in all intervals Q_{2n} by setting

$$t_{2n}(x) = \begin{cases} [2n, 2n+s), & \text{if } x = [0, s), \\ [2n+s, 2n+1), & \text{if } x = [s, 1), \\ [2n, 2n+s) \cup [2n+t, 2n+1), & \text{if } x = [0, s) \cup [t, 1). \end{cases} \tag{40}$$

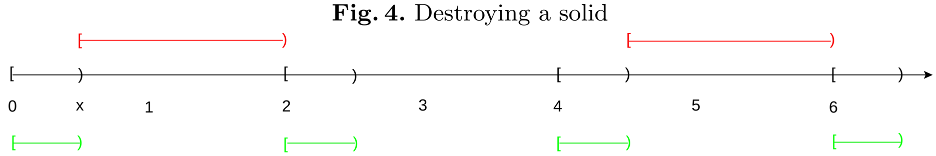
Fig. 3. Some holes and solids in $IntAlg([0,1))$



to obtain **x**.
2. Fill the space between $t_{4n}(x)$ and $t_{4n+2}(x)$ to obtain a hole of **x**.

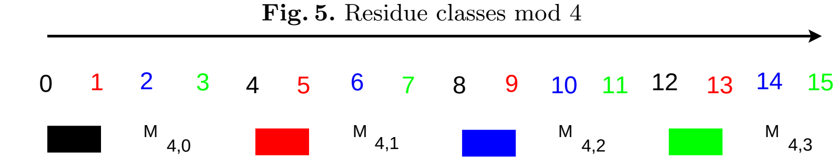
see Figure 4.

Fig. 4. Destroying a solid



For $2 \leq n,\ 0 \leq k \lneq n$, let $M_{n,k} = \{q \in \omega : q \equiv k \mod n\}$ be the k^{th}–residue class of n, see Figure 5. B^+ is the collection of all sets of the form $M_{n,k_0} \cup \ldots \cup M_{n,k_t}$. Here, as below, we suppose that $0 \leq k_0 \lneq k_1 \lneq \ldots \lneq k_t \lneq n$. Furthermore, set $B = B^+ \cup \emptyset$. Then,

Fig. 5. Residue classes mod 4



Proposition 9. *B is an atomless Boolean subalgebra of 2^ω.*

For each $n \in \omega$ let $t_n : \mathbb{Q} \to \mathbb{Q}$ be the translation $q \mapsto q + n$. Furthermore, we let L_n be the interval algebra of the rational interval $[n, n+1)$ (with $n+1$ taking the place of ∞) endowed with the contact relation defined in (35), and define $f_n : L_0 \to L_n$ by $f_n(x) = \{t_n(q) : q \in x\}$. Clearly, f_n is a BCA isomorphism, and $n \neq m$ implies $f_n(x) \cap f_m(y) = \emptyset$. Next, define $g : B \times L_0 \to 2^{\mathbb{Q}}$ by $g(T, x) = \bigcup\{f_n(x) : n \in T\}$. Note that $g(T, x) = \emptyset$ if $T = \emptyset$ or $x = \emptyset$. Furthermore,

Lemma 5. *1.* $g(T,x) \cup g(T,y) = g(T,x+y)$.
2. $g(T,x) \cup g(S,x) = g(S \cup T, x)$.
3. $-g(T,x) = g(-T,[0,1)) \cup g(T,x^*)$.
4. $g(T,x) \cap g(S,y) = g(T \cap S, x \cdot y)$.

Let A be the collection of all finite unions of sets of the form $g(T,x)$, where $T \subseteq \omega$ and $x \in L_0$. Then,

Proposition 10. *A is a Boolean subalgebra of $2^{\mathbb{Q}}$.*

Observe that each nonempty $a \in A$ is a union of intervals of $\mathbb{Q}$, and thus, the notion $\mathrm{rel}(a)$ still makes sense. As with $IntAlg(L)$, we define C on A^+ by

$$aCb \iff (a \cap b) \cup (\mathrm{rel}(a) \cap \mathrm{rel}(b)) \neq \emptyset.$$

Now,

Proposition 11. *$\langle A, C\rangle$ is an RCC algebra which has no proper solid regions.*

This construction has interesting topological consequences. We know from [13], that $\langle A, C\rangle$ is isomorphic to a dense substructure of the Boolean algebra $\mathrm{RegCl}(X)$ of regular closed sets of some connected T_1 space X. Now suppose that $\langle A, C\rangle$ itself is such a substructure of some such space X. Since A is dense in $\mathrm{RegCl}(X)$, its non-zero elements are a basis for the closed sets. Since for each $x \in A^+$, its complement x^* in $\mathrm{RegCl}(X)$ is topologically disconnected, we observe that X has an open basis of disconnected regular open sets.

Some more general constructions of BCAs from a given BCA can be summarized as follows:

Proposition 12. *[14]*

1. *If $\langle A, C\rangle$ is a BCA and A is a dense subalgebra of B, then there is a contact relation D on B such that $D \restriction A^2 = C$.*
2. *If $\langle B, C\rangle$ is a BCA and A a dense subalgebra of B, the $C \restriction (A \times A)$ is a contact relation on A.*
3. *Let B be atomless, F, G be distinct ultrafilters of B, and $R = C \cup (F \times G) \cup (G \times F)$. Then, R is a contact relation on B. Furthermore, if C satisfies C7, so does R.*

5.3 Representability

The major problem in mereotopology is how algebraic properties of BCAs and their extensions translate into topological ones (and vice versa), and whether the axioms are complete for certain classes of topological spaces. In other words, can we find topological properties such that the spaces with these properties and standard topological contact are completely characterized by a class of BCAs? To make this clearer, we say that a BCA $\langle B, C\rangle$ is *representable* if there are a topological space $\langle X, \tau\rangle$ and a Boolean embedding $e : B \to \text{RegOp}(X)$ such that $aCb \iff e(a)C_\tau e(b)$ for all $a, b \in B$ and the standard topological contact C_τ on X as defined in (10).

The proximity approach The first result in this direction was the work of [44], who used proximity techniques to show that every BCA B satisfying C6 has a representation in a compact Hausdorff space X. If B is, in addition an RCC model, then X is connected in the topological sense. Applications of proximity spaces to similar problems can be found in [42] and [8]. There, proximity spaces are used to formalise the notions of local and global similarity relations. A local similarity relation has a semantics just as the overlap relation O in mereology, and a global similarity relation is interpreted just by the proximity relation. This shows another possible approach to the theory of mereological relations – the theory of similarity relations (or, more generally, informational relations) in information systems (see [41] for references). As the axioms for proximity spaces and BCAs have a common core, the following comes as no surprise:

Lemma 6. *[44] Let (X, δ) be a separated proximity space, and τ be the topology on X defined by* (8)*. Then, $(RC(X), \delta)$ is a BCA which satisfies C6.*

$(RC(X), \delta)$ is called the *proximity connection algebra over* (X, δ). It is called a *standard connection algebra*, if

$$A\delta B \text{ iff } A \cap B \neq \emptyset. \tag{41}$$

It can be shown that

Proposition 13. *[44] Each proximity connection algebra is isomorphic to a standard connection algebra.*

The proof depends on the Smirnov Compactification Theorem [26] which, in turn, makes heavy use of C6. It can now be shown that each BCA satisfying C6 can be isomorphically embedded into a proximity connection algebra and thus, into a standard proximity connection algebra. The strategy is analogous to Stone's representation Theorem for Boolean algebras which uses sets of ultrafilters to determine the points of the algebra. In the present case, the notion of $p-cluster$ is borrowed from the theory of proximity spaces; in the rest of this Section, we will suppose that $\langle B, C\rangle$ is a BCA satisfying C6.

A nonempty subset Γ of B is called a *clan* if the following conditions are satisfied:

CL1. If $x, y \in \Gamma$ then xCy.
CL2. If $x + y \in \Gamma$ then $x \in \Gamma$ or $y \in \Gamma$.
CL3. If $x \in \Gamma$ and $x \leq y$, then $y \in \Gamma$.

A clan is called a *p–cluster* if it satisfies

CL4. If xCy for every $y \in \Gamma$, then $x \in \Gamma$.

The set of all p–clusters on B is denoted by $\mathrm{pClust}(B)$. In analogy to the Stone representation theorem for Boolean algebras, we define a mapping $h : B \to 2^{\mathrm{pClust}(B)}$ by

$$h(a) = \{\Gamma \in \mathrm{pClust}(B) : a \in \Gamma\}. \tag{42}$$

We observe that in this context, the p–clusters play the role that ultrafilters play in Stone's Theorem, i.e. they are the points of our representation space. Next, we set for $X, Y \subseteq \mathrm{pClust}(B)$

$$X \delta_B Y \text{ iff } (\forall x, y \in B)[x \in \bigcap X \text{ and } y \in \bigcap Y \Longrightarrow xCy].$$

By definition of h, we have

$$X \delta_B Y \Longleftrightarrow (\forall x, y \in B)[X \subseteq h(x) \text{ and } Y \subseteq h(y) \Longrightarrow xCy].$$

Let $\mathrm{PS}(B)$ be the structure $\langle \mathrm{pClust}(B), \delta_B \rangle$. Using the the Smirnov Compactification Theorem [26], it can be shown that

Proposition 14. *[44]* $\mathrm{PS}(B)$ *is a separated proximity space.*

Let X be the powerset of $\mathrm{pClust}(B)$, and τ be the topology on X induced by δ_B, as defined in (8). Then, for each $M \in X$ we have

$$\mathrm{cl}(M) = \{\Gamma \in \mathrm{pClust}(B) : (\forall x, y \in B)[x \in \Gamma \text{ and } M \subseteq h(y) \Longrightarrow xCy]\}.$$

With these instruments, the following representation Theorem can now be proved:

Proposition 15. *[44]*

1. *Each connection algebra can be embedded into a proximity connection algebra.*
2. *Each connection algebra can be embedded into a standard connection algebra.*

So, the representation problem has a solution in case the BCA satisfies the interpolation axiom C6. The next section will show how to remove this restriction.

Representing BCAs The definition of p–cluster was very much geared towards the interpolation axiom C6. In order to remove the requirement of C6, one can look at a more general definition of p–cluster: Clearly, the class of clans on B is closed under union of chains, and thus each clan is contained in a maximal element which we simply call *cluster*. The set of all clusters in B will be denoted by $\operatorname{Clust} B$. Clearly, each p–cluster is a cluster, but the converse need not be true. However, if C satisfies C6, then the two notions coincide:

Proposition 16. *[13] Suppose that C satisfies C6. Then, each cluster is a p–cluster.*

With this more general notion of cluster, we can proceed along the lines of [44]: Let $\langle B, C\rangle$ be a BCA, $X = \operatorname{Clust} B$ and $h : B \to 2^X$ be defined by the Stone–like assignment $h(x) = \{\Gamma \in \operatorname{Clust} B : x \in \Gamma\}$. Then, h is injective and preserves $+$:

Lemma 7. *1. $x \leq y \iff h(x) \subseteq h(y)$.*
2. $h(0) = \emptyset,\ h(1) = X$, and $h(x) \cup h(y) = h(x + y)$ for all $x, y \in B$.

Using this embedding, it is now possible to show

Proposition 17. *1. Each BCA B is isomorphic to a dense substructure of some $\langle \operatorname{RegCl}(X), C_\tau\rangle$ where τ is T_1 and weakly regular.*
2. B satisfies C7 if and only if X is connected.

The following example shows that $\langle X, \tau\rangle$ need not be a T_2 space:

Example 1. Let $\langle B, C\rangle$ be the interval algebra of Proposition 8, $0 \lneq a \lneq b \lneq c \lneq 1$ and F_a, F_b, F_c be the ultrafilters of B of all elements of B containing, respectively, a, b or c. Let $D = C \cup (F_a \times F_b) \cup (F_b \times F_a) \cup (F_a \times F_c) \cup (F_c \times F_a)$. By Proposition 12, D is a contact relation. One can show that $\Gamma = F_a \cup F_b$ and $\Delta = F_a \cup F_c$ are clusters. Incidentally, this shows that an ultrafilter can be contained in two different clusters which is not possible for p-clusters. Assume that there are open sets u, v such that $\Gamma \in u$, $\Delta \in v$ and $u \cap v = \emptyset$. Since the sets $h(x)$ are a basis for the closed sets, there are x, y in B such that $\Gamma \notin h(x)$, $\Delta \notin h(y)$ and $h(x) + h(y) = X$. Since h is an embedding, the latter implies $x + y = 1$. On the other hand, $\Gamma \notin h(x)$ implies that $a \notin x$ and $b \notin x$, and $\Delta \notin h(y)$ implies that $a \notin y$ and $c \notin y$. Together, we obtain $a \notin x + y$, contradicting $x + y = 1$. □

This also shows that not every RCC–algebra can be represented in a regular T_1 space (which is necessarily T_2). This shows that the assumption of regularity as standard models for RCC algebras is too strong: Not every RCC algebra has a representation in a regular T_1 space. Weak regularity, however, is necessary and sufficient.

6 Summary and outlook

We have collected various results concerning the construction of Boolean Contact Algebras and their mereotopological expressiveness. The most striking example of BCA was the "hole algebra" of Proposition 11; it shows that BCAs indeed give rise to quite strange worlds of regions. A basic open problem in this area is the following:

– Is it possible to define a contact relation which satisfies C7 on every atomless BA?

The question of representability of RCC algebras, which has been an open problem for some time, was answered by Proposition 17; Example 1 tells us that this result is the best possible. In particular, the RCC calculus is too weak to include connection substructure of regular closed algebras of regular T_1 spaces. So, one can ask

– Which other important topological properties can be expressed by additional axioms?

More concretely, one can ask

– What is the topological expressive power of the first order theory of specific BCAs, such as the polygonal algebras or the bounded–cobounded algebras [29]?

Another area of interest is the the study of the collection of contact relations on a Boolean algebra. The interesting case here is when the underlying BA is atomless (see Proposition 3).

– Investigate lattice theoretic properties of the collection of contact relations on an atomless Boolean algebra.

Here, one may want to obtain general results, as well as some insight for concrete BAs, such as the regular closed algebra of the line or the plane.

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